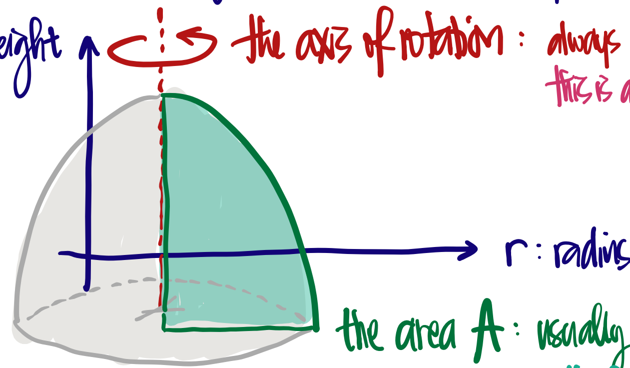
Methods for Calculating the Volume of Solids of Revolution.

In this case, the solid can be characterized by an area on the radius-height plane like so:

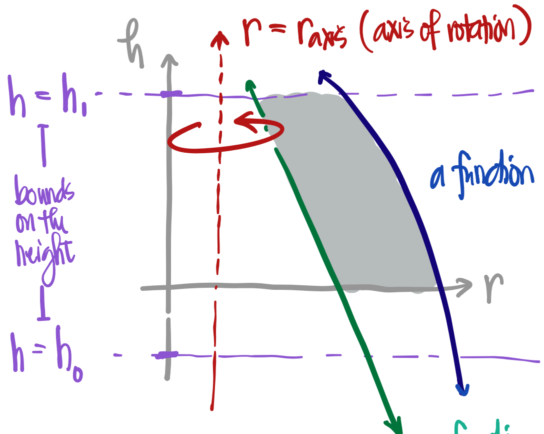


the axis of rotation : always parallel to the height axis.
this is also usually the height axis, $r=0$.

the area A : usually bounded by some number of functions
in the form $r(h)=(\cdots)$, i.e. height h is variable
or $h(r)=(\cdots)$, i.e. radius r is variable.

Method 1. Washer Method

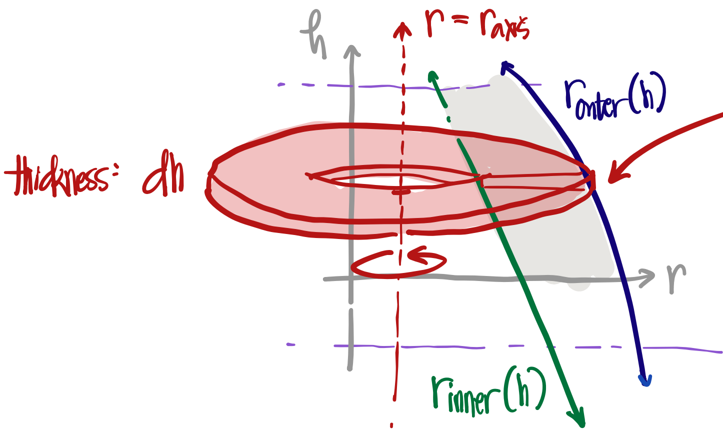
Assume that the area on the radius-height plane is bounded like so:



a function $r_{outer}(h)$ in terms of the height h.

a function $r_{inner}(h)$ in terms of the height h.

We slice the solid perpendicular to the height axis: i.e., we integrate with respect to h.



This washer has volume:

$$dV = dV_{outer\ disk} - dV_{inner\ disk}$$
$$= \pi\left(r_{outer} - r_{axis}\right)^2 dh - \pi\left(r_{inner} - r_{axis}\right)^2 dh$$
$$= \pi\left[\left(r_{outer} - r_{axis}\right)^2 - \left(r_{inner} - r_{axis}\right)^2\right] dh$$

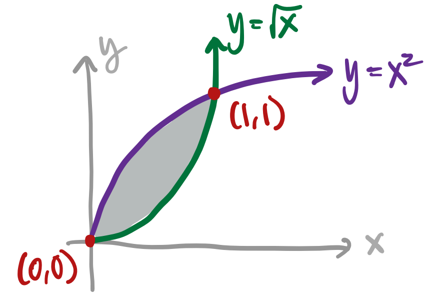
The volume of the solid is given by:

$$V = \pi\int_{h_0}^{h_1}\left[\left(r_{outer}(h) - r_{axis}\right)^2 - \left(r_{inner}(h) - r_{axis}\right)\right] dh$$

with

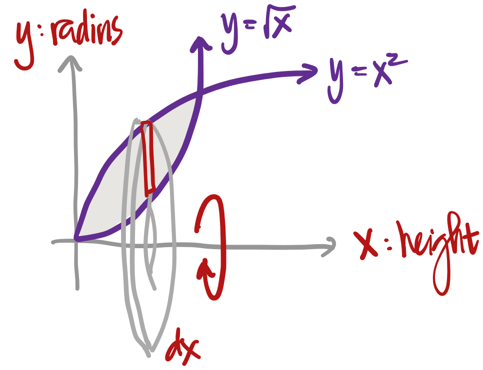
axis of rotation : $r = r_{axis}$
bounds on height : $h \in [h_0, h_1]$
outer function : $r_{outer}(h)$
inner function : $r_{inner}(h)$

Example 1. Let A be the region bounded by the curves $y=x^2$ and $y=\sqrt{x}$.



Part (a). Find the volume V_a of the solid formed by revolving the region A about the x-axis.

Solid:



! Here, the x-axis is the height axis and the y-axis is the radius axis.

To use the Shell Method, we want to integrate with respect to x. i.e. the height axis.

Required info:

axis of rotation: $y=0$

bounds on height: $x \in [0,1]$

outer radius: $y_{outer} = \sqrt{x}$ (the curve furthest from the axis)

inner radius: $y_{inner} = x^2$

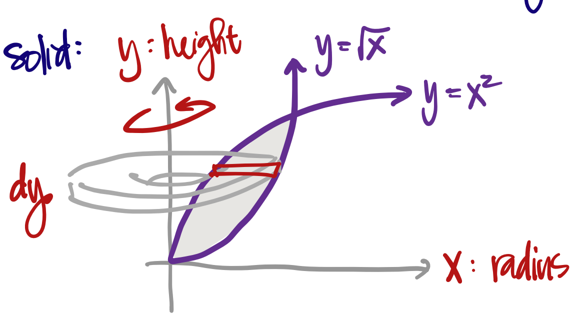
Then, $$V_a = \int_{y=0}^{y=1} \pi\left[(y_{outer}-0)^2 - (y_{inner}-0)^2\right] dx$$

$$= \pi\int_0^1 (\sqrt{x})^2 - (x^2)^2\, dx = \pi\int_0^1 x - x^4\, dx = \pi\left[\tfrac{1}{2}x - \tfrac{1}{5}x^5\right]_0^1$$

$$= \pi\left[\tfrac{1}{2} - \tfrac{1}{5}\right] = \frac{3\pi}{10};$$

Part (b). Find the volume V_b of the solid formed by revolving the region A about the y-axis.

Solid:



! The y-axis is the height axis.

To use the Shell Method, we want to integrate with respect to y.

Required Info: axis of rotation: $x=0$

bounds on height: $y \in [0,1]$

outer radius: $y = x^2$; $x = \pm\sqrt{y}$; Since $y \in [0,1]$, $x = \sqrt{y}$

inner radius: $y = \sqrt{x}$; $x = y^2$;

Then,

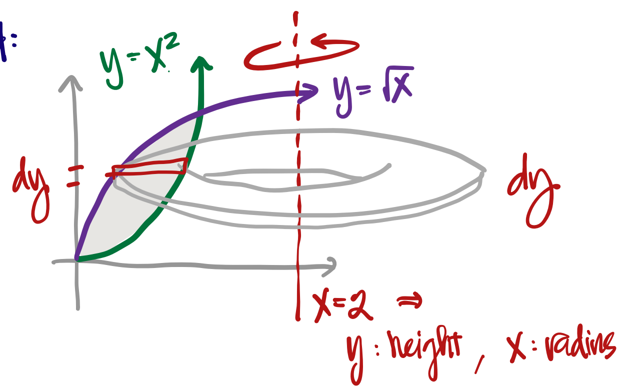
$$V_b = \pi \int_{y=0}^{y=1} (x_{outer} - 0)^2 - (x_{inner} - 0)^2 \, dy$$

$$= \pi \int_0^1 (\sqrt{y})^2 - (y^2)^2 \, dy = \pi \int_0^1 y - y^4 \, dy =$$

$$= \pi \left[\frac{1}{2}y - \frac{1}{5}y^5 \right]_0^1 = \pi \left(\frac{1}{2} - \frac{1}{5} \right) = \frac{3\pi}{10} ;$$

Part (c). Find the volume V_c of the solid formed by revolving A about $x=2$.

solid:



Info: axis of rotation : $x=2$

bounds on height : $y \in [0,1]$

outer function, x_{outer} : $y = \sqrt{x}$, $x = y^2$

lower function, x_{inner} : $y = x^2$, $x = \sqrt{y}$

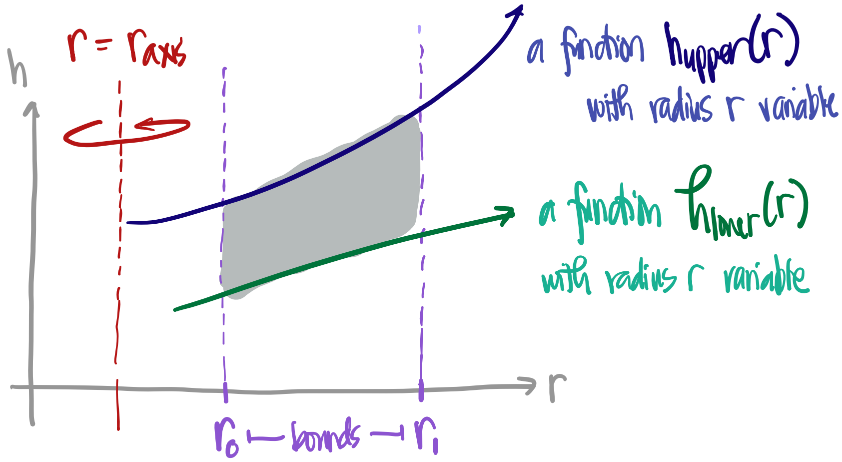
Then, $V_c = \pi \int_{y=0}^{y=1} (2 - x_{outer})^2 - (2 - x_{inner})^2 \, dy$ ← We have $(2-x_{outer})^2$ instead of $(x_{outer}-2)^2$ since x_{outer} is on the left of $x=2$. In any case, $(2-x_{outer})^2 = (x_{outer}-2)^2$

$$= \pi \int_0^1 (2 - y^2)^2 - (2 - \sqrt{y})^2 \, dy$$

$$= \cdots = \pi \left(\frac{31}{30} \right) = \frac{31}{30}\pi ;$$

Method 2. Cylindrical Shells Method

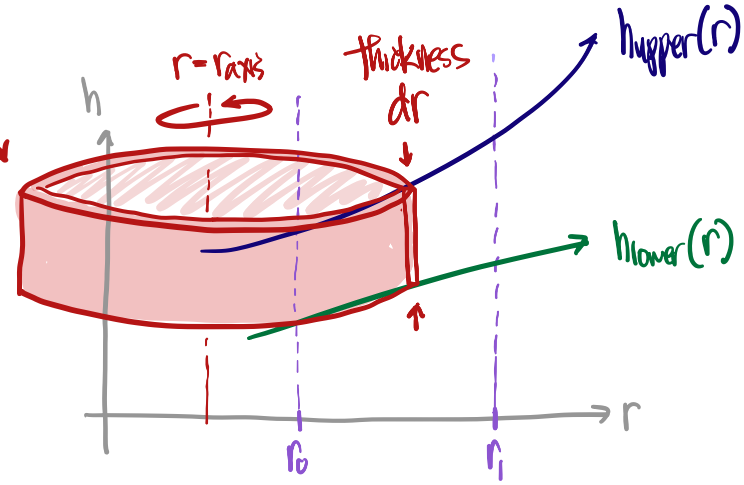
Assume that the area on the radius-height plane is bounded like so:



We slice the solid **parallel** to the axis of rotation, i.e. we integrate with respect to r.

This hollow cylinder has volume

$$dV = (\text{circumference})(\text{height})(\text{thickness})$$
$$= (r - r_{axis})(h_{upper} - h_{lower})\, dr$$



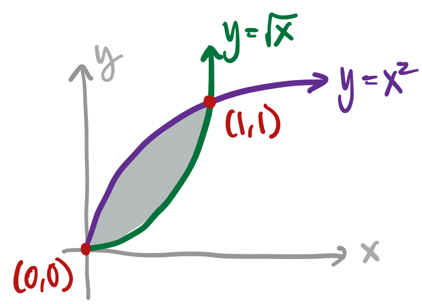
Then, the volume of the solid is given by:

$$V = \int_{r_0}^{r_1} 2\pi (r - r_{axis}) \left[h_{upper}(r) - h_{lower}(r) \right] dr$$

given

axis of rotation	:	$r = r_{axis}$
bounds on radius	:	$r \in [r_0, r_1]$
upper function	:	$h = h_{upper}(r)$
lower function	:	$h = h_{lower}(r)$

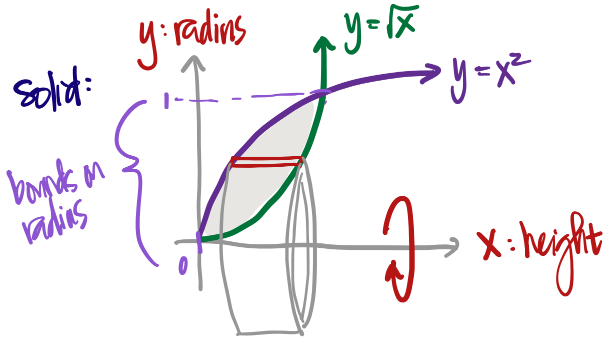
Example 2. Let A be the region bounded by the curves $y = x^2$ and $y = \sqrt{x}$.



(Same problem but solved with the Cylindrical Shells Method)

Part (a). Find the volume V_a of the solid formed by revolving the region A about the x-axis.

solid:



Since the y-axis is the radius axis, we want to integrate with respect to y.

Then,

axis of rotation	: $y=0$	
bounds on radius	: $y \in [0,1]$	
upper function, x_{upper}	: $y = x^2$	, $x = \sqrt{y}$
lower function, x_{lower}	: $y = \sqrt{x}$	, $x = y^2$

$$V_a = \int_{y=0}^{y=1} 2\pi(y-0)\left(x_{upper}(y) - x_{lower}(y)\right)\,dy$$

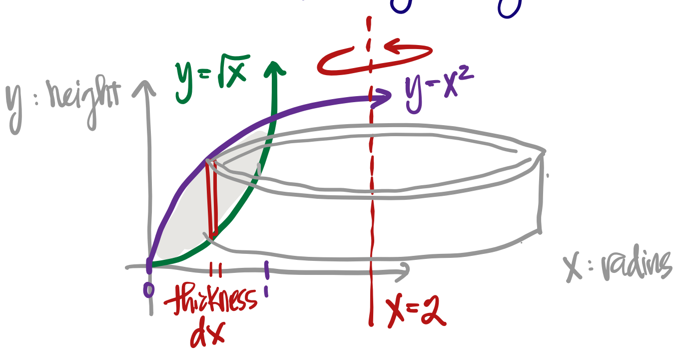
$$= 2\pi \int_0^1 y\left(\sqrt{y} - y^2\right)dy = 2\pi \int_0^1 y^{\frac{3}{2}} - y^3\,dy$$

$$= 2\pi\left[\frac{2}{5}y^{\frac{5}{2}} - \frac{1}{4}y^4\right]_0^1 = 2\pi\left[\frac{2}{5} - \frac{1}{4}\right] = 2\pi\left(\frac{3}{20}\right) = \frac{3\pi}{10};$$

Part (b). —

Part (c). Find the volume V_c of the solid formed by revolving A about $x=2$.

solid:



Then,

axis of rotation	: $x=2$	;	upper function, y_{upper} :	$y=x^2$
bounds on radius	: $x \in [0,1]$		lower function, y_{lower} :	$y=\sqrt{x}$

$$V = \int_{x=0}^{x=1} 2\pi(r - r_{axis})\left(y_{upper}(x) - y_{lower}(x)\right)dx$$

$$= 2\pi \int_0^1 (x-2)\left(x^2 - \sqrt{x}\right)dx = \dots = 2\pi\left(\frac{31}{60}\right) = \frac{31}{30}\pi;$$